



Universidad
Carlos III de Madrid



This is a postprint version of the following published document:

Marcellán, F.; Ranga, A. S. (2014). "Sieved para-orthogonal polynomials on the unit circle". *Applied Mathematics and Computation*, v. 244, pp. 335-343.
DOI: 10.1016/j.amc.2014.07.014

© Elsevier 2014



This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.

Sieved para-orthogonal polynomials on the unit circle ^{*}

F. Marcellán ^a, A. Sri Ranga ^{b, *}

^aInstituto de Ciencias Matemáticas (ICMAT) and Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain

^bDepartamento de Matemática Aplicada, IBILCE, UNESP – Universidade Estadual Paulista, 15054-000 São José do Rio Preto, SP, Brazil

Key words

Sieved orthogonal polynomials on the unit circle
Para-orthogonal polynomials
Chain sequences

Abstract

Sieved orthogonal polynomials on the unit circle were introduced independently by Ismail and Li (1992) [15] and Marcellán and Sansigre (1991) [19]. We look at the para-orthogonal polynomials, chain sequences and quadrature formulas that follow from the kernel polynomials of sieved orthogonal polynomials on the unit circle.

1. Introduction

Given a nontrivial positive measure $\mu(\zeta) = \mu(e^{i\theta})$ supported on the unit circle $\mathcal{C} = \{\zeta = e^{i\theta} : 0 \leq \theta \leq 2\pi\}$, it is well known that the associated sequence of monic OPUC (orthogonal polynomials on the unit circle) $\{S_n(z)\}_{n=0}^{\infty}$ can be defined by

$$\int_{\mathcal{C}} \zeta^j S_n(\zeta) d\mu(\zeta) = \int_0^{2\pi} e^{ij\theta} S_n(e^{i\theta}) d\mu(e^{i\theta}) = 0, \quad 0 \leq j \leq n-1, \quad n \geq 1.$$

Letting $\kappa_n^2 = \|S_n\|^2 = \int_{\mathcal{C}} |S_n(\zeta)|^2 d\mu(\zeta)$, the orthonormal polynomials on the unit circle are $s_n(z) = \kappa_n^{-1} S_n(z)$, $n \geq 0$.

OPUC were introduced by Gábor Szegő in the first half of the 20th century (see the monograph [28]). Thus, they are also referred to as Szegő polynomials. These polynomials, which have received a lot of attention in recent years (see, for example, [1,5,9,8,18,21, 23,27,29]), have applications in quadrature rules, signal processing, operator and spectral theory and many other topics.

Chapter 8 of Ismail's recent book [14] on these polynomials and the two recent volumes [24,25] by Barry Simon, specifically under the title “orthogonal polynomials on the unit circle”, have provided us with many useful tools for further research in this area.

The monic OPUC satisfy the so called forward and backward recurrence relations, respectively,

$$\begin{aligned} S_n(z) &= zS_{n-1}(z) - \bar{\alpha}_{n-1}S_{n-1}^*(z), \\ S_n(z) &= (1 - |\alpha_{n-1}|^2)zS_{n-1}(z) - \bar{\alpha}_{n-1}S_n^*(z), \quad n \geq 1, \end{aligned} \tag{1.1}$$

where $\alpha_{n-1} = \overline{S_n(0)}$ and $S_n^*(z) = z^n \overline{S_n(1/\bar{z})}$ denotes the reversed (reciprocal) polynomial of $S_n(z)$. Following Simon [24], we refer to the numbers α_n as Verblunsky coefficients. It is known that these coefficients are such that $|\alpha_n| < 1$, $n \geq 0$, as well as that OPUC and the associated measure are completely characterized by the Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ as stated by the following theorem.

^{*} The first author has been supported by Dirección General de Política Científica y Tecnológica, Ministerio de Economía y Competitividad of Spain, Grant MTM2012-36732-C03-01 as well as by CAPES, Brazil. The second author's work was supported by funds from CAPES, CNPq and FAPESP of Brazil.

* Corresponding author.

E-mail address: ranga@ibilce.unesp.br (A. Sri Ranga).

Theorem A. Given an arbitrary sequence of complex numbers $\{\alpha_n\}_{n=0}^{\infty}$, where $|\alpha_n| < 1$, $n \geq 0$, then, associated with this sequence, there exists a unique nontrivial probability measure μ on the unit circle such that the polynomials $\{S_n(z)\}_{n=0}^{\infty}$ generated by (1.1) are the respective OPUC.

Here, μ is a nontrivial positive measure if its support is infinite and it is a nontrivial probability measure if $\mu_0 \int_{\mathbb{C}} d\mu(\zeta) = 1$. The above theorem, known as Favard's theorem for the unit circle, has been referred to as Verblunsky's theorem in Simon [24].

Given the sequence of Verblunsky coefficients $\{\alpha_n\}_{n=0}^{\infty}$ let μ be the associated nontrivial probability measure and let $\{S_n(z)\}_{n=0}^{\infty}$ be the corresponding OPUC. For a positive integer ℓ the sieved OPUC $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ are defined as those orthogonal polynomials associated with the Verblunsky coefficients $\{\alpha_n^{(\ell)}\}_{n=0}^{\infty}$ given by

$$\alpha_n^{(\ell)} = \begin{cases} 0, & \text{if } (n+1) \not\equiv 0 \pmod{\ell}, \\ \alpha_{[n/\ell]}, & \text{if } (n+1) \equiv 0 \pmod{\ell}, \end{cases} \quad (1.2)$$

for $n \geq 0$. We also denote by $\mu^{(\ell)}$ the nontrivial probability measure on the unit circle associated with $\{\alpha_n^{(\ell)}\}_{n=0}^{\infty}$.

Note that $\{S_n^{(1)}(z)\}_{n=0}^{\infty}$ are the polynomials $\{S_n(z)\}_{n=0}^{\infty}$. The earliest treatment of the sieved orthogonal polynomials $\{S_n^{(\ell)}(z)\}_{n=0}^{\infty}$ for $\ell \geq 2$ is found in Ismail and Li [15]. However, the sieved orthogonal polynomials $\{S_n^{(2)}(z)\}_{n=0}^{\infty}$ have been studied earlier than in [15] by Marcellán and Sansigre (see [19,20]). The following results, established in [15], will be the basic requirement for the results obtained in the present manuscript.

$$S_{r\ell+j}^{(\ell)}(z) = z^j S_r^{(\ell)}(z^\ell), \quad j = 0, 1, \dots, \ell-1, \quad r \geq 0, \quad (1.3)$$

and

$$d\mu^{(\ell)}(e^{i\theta}) = \ell^{-1} d\mu(e^{i\ell\theta}), \quad 0 \leq \theta \leq 2\pi.$$

For the reversed polynomials $S_n^{(\ell)*}(z)$ there hold

$$S_{r\ell+j}^{(\ell)*}(z) = z^{r\ell+j} \overline{S_{r\ell+j}^{(\ell)}(1/z)} = (z^\ell)^r \overline{S_r^{(\ell)}(1/z^\ell)} = S_r^*(z^\ell), \quad j = 0, 1, \dots, \ell-1, \quad r \geq 0. \quad (1.4)$$

The aim of the present manuscript is to explore the connection between para orthogonal polynomials, chain sequences and quadrature formulas that follow from the kernel polynomials of the sieved OPUC. The structure of the manuscript is as follows. In Section 2 we present a basic background concerning para orthogonal polynomials associated with a nontrivial probability measure on the unit circle, the three term recurrence relation they satisfy, the representation of the coefficients of such a recurrence relation as chain sequences and the role of zeros of para orthogonal polynomials in Gaussian quadrature rules on the unit circle. In Section 3, we consider para orthogonal polynomials associated with the sieved measure on the unit circle. Then we obtain its corresponding three term recurrence relation as well as the chain sequences for the coefficients of such a recurrence relation. Finally, in Section 4 we deal with the Gaussian quadrature rules for such sieved para orthogonal polynomials. The nodes and the corresponding weights are deduced.

2. Para-orthogonal polynomials from kernel polynomials

Recall that the Christoffel–Darboux formula of order n associated with the sequence $\{S_n(z)\}_{n=0}^{\infty}$ of OPUC is such that

$$K_n(z, w) = \sum_{j=0}^n \overline{s_j(w)} s_j(z) = \frac{S_{n+1}^*(w) S_{n+1}^*(z)}{1 - \overline{w} z}.$$

Here, $s_n(z) = \kappa_n S_n(z)$ are the normalized OPUC. See, for example [24, Thm. 2.2.7], where $K_n(z, w)$ is referred to as a CD kernel, meaning a Christoffel–Darboux kernel.

With $|w| = 1$, we consider the sequence $\{P_n(w; z)\}_{n=0}^{\infty}$ of polynomials in z given by

$$P_n(w; z) = \frac{\kappa_{n+1}^2 \overline{w}}{S_{n+1}(w)} \frac{K_n(z, w)}{1 + \tau_{n+1}(w) \alpha_n}, \quad n \geq 0,$$

where $\tau_n = S_n(w)/S_n^*(w)$, $n \geq 0$. It is easily verified that $P_n(w; z)$ is a monic polynomial of degree n in z , which can be simply written as

$$P_n(w; z) = \frac{1}{z} \frac{S_{n+1}(z)}{w} \frac{\tau_{n+1}(w) S_{n+1}^*(z)}{1 + \tau_{n+1}(w) \alpha_n} = \frac{1}{z} \frac{1}{w} [z S_n(z) - \tau_n(w) S_n^*(z)], \quad n \geq 0. \quad (2.1)$$

Since $|w| = 1$ we have $|\tau_n(w)| = 1$ for $n \geq 0$. Hence, $S_{n+1}(z) - \tau_{n+1}(w) S_{n+1}^*(z)$ is known as a para orthogonal polynomial associated with S_{n+1} . From known properties of para orthogonal polynomials (see [17]), $S_{n+1}(z) - \tau_{n+1}(w) S_{n+1}^*(z)$ has $n+1$ simple zeros on the unit circle $|z| = 1$. In particular, w is one of the zeros of $S_{n+1}(z) - \tau_{n+1}(w) S_{n+1}^*(z)$. Consequently, the polynomial $P_n(w; z)$ has all its n zeros simple and lying on the unit circle $|z| = 1$. However, none of the zeros of $P_n(w; z)$ can be equal to the value w .

Perhaps the first reference that explicitly brings the connection between CD kernels and para orthogonal polynomials is González Vera et al. [12] (see also [3, Thm. 2.1]). Such results in a setting based on linear algebra and also without the use of the name para orthogonal appear, even earlier than in [12], in Gragg [13]. However, the name para orthogonal polynomials for $S_n(z) = \tau_n S_n^*(z)$, where $|\tau_n| = 1$ and S_n are OPUC, is due to Jones et al. [17]. We may refer to the polynomials $(z - w)P_n(w; z)$ as the CD kernel POPUC.

Concerning contributions emphasizing the connection between CD kernels and para orthogonal polynomials we refer to [4, 11], published independently in the same year. These papers bring many interesting results concerning the zeros of para orthogonal polynomials such as interlacing, separation and asymptotic distribution. For a more recent paper in this direction we cite [30]. We also cite [26], where there is a nice section on CD kernels and para orthogonal polynomials.

Positive chain sequences play an important role in this manuscript. Following the definition adopted by Chihara [7], we say that $\{d_n\}_{n=1}^\infty$ is a positive chain sequence if there exists a second sequence $\{g_n\}_{n=0}^\infty$ such that

$$\begin{aligned} (i) \quad & 0 \leq g_0 < 1, \quad 0 < g_n < 1 \quad \text{for } n \geq 1; \\ (ii) \quad & d_n = (1 - g_{n-1})g_n \quad \text{for } n \geq 1. \end{aligned} \quad (2.2)$$

The sequence $\{g_n\}_{n=0}^\infty$ is called a parameter sequence of $\{d_n\}_{n=1}^\infty$. Every positive chain sequence has a minimal parameter sequence $\{m_n\}_{n=0}^\infty$ such that

$$m_0 = 0, \quad m_n = d_n / (1 - m_{n-1}) \quad \text{for } n \geq 1,$$

and a maximal parameter sequence $\{M_n\}_{n=0}^\infty$ (some times can be the same as the minimal) such that if $M_0 < g_0 < 1$ then the sequence $\{g_n\}_{n=0}^\infty$ generated by $g_n = d_n / (1 - g_{n-1})$, $n \geq 1$, does not satisfy (i) of (2.2).

Some of the results shown recently in [8] that are relevant for the present manuscript can be summarized as follows.

For the (special and appropriately scaled) CD kernel POPUC $(z - 1)P_n(1; z)$, given by

$$R_n(z) = \sigma_n \frac{z S_n(z) - \tau_n S_n^*(z)}{z - 1}, \quad n \geq 0, \quad (2.3)$$

where

$$\begin{aligned} \tau_0 &= \frac{S_0(1)}{S_0^*(1)} = 1 \quad \text{and} \quad \tau_n = \frac{S_n(1)}{S_n^*(1)} = \frac{\tau_{n-1} \bar{\alpha}_{n-1}}{1 - \tau_{n-1} \alpha_{n-1}}, \quad n \geq 1, \\ \sigma_0 &= 1 \quad \text{and} \quad \sigma_n = \frac{1 - \tau_{n-1} \alpha_{n-1}}{1 - \operatorname{Re}(\tau_{n-1} \alpha_{n-1})} \sigma_{n-1}, \quad n \geq 1, \end{aligned}$$

the following three term recurrence formula holds.

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1, \quad (2.4)$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$, where the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_{n+1}\}_{n=1}^\infty$ are such that

$$\begin{aligned} c_n &= \frac{\operatorname{Im}(\tau_{n-1} \alpha_{n-1})}{1 - \operatorname{Re}(\tau_{n-1} \alpha_{n-1})} = i \frac{\tau_n - \tau_{n-1}}{\tau_n + \tau_{n-1}}, \\ d_{n+1} &= \frac{1}{4} \frac{[1 - |\tau_{n-1} \alpha_{n-1}|^2][1 - \tau_n \alpha_n]^2}{[1 - \operatorname{Re}(\tau_{n-1} \alpha_{n-1})][1 - \operatorname{Re}(\tau_n \alpha_n)]}, \quad n \geq 1. \end{aligned} \quad (2.5)$$

Moreover, $\{d_{n+1}\}_{n=1}^\infty$, where $d_{n+1} = (1 - m_n)m_{n+1}$, is a positive chain sequence with the parameter sequence $\{m_{n+1}\}_{n=0}^\infty$ given by

$$m_{n+1} = \frac{1}{2} \frac{|1 - \tau_n \alpha_n|^2}{[1 - \operatorname{Re}(\tau_n \alpha_n)]}, \quad n \geq 0.$$

Since $0 < m_1 < 1$, setting

$$d_1 = m_1 = \frac{1}{2} \frac{|1 - \tau_0 \alpha_0|^2}{[1 - \operatorname{Re}(\tau_0 \alpha_0)]}, \quad (2.6)$$

then $\{m_n\}_{n=0}^\infty$, with $m_0 = 0$, is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^\infty$.

The sequence $\{m_n\}_{n=0}^\infty$, together with the sequence $\{c_n\}_{n=1}^\infty$, can be used to characterize the above measure μ . For example, the associated monic OPUC $\{S_n(z)\}_{n=0}^\infty$ can be given as

$$S_n(z) = \prod_{k=1}^n (1 + ic_k) R_n(z) = 2(1 - m_n)R_{n-1}(z), \quad n \geq 1,$$

and the corresponding Verblunsky coefficients can be derived from

$$\tau_n = \frac{1 - ic_n}{1 + ic_n} \tau_{n-1} \quad \text{and} \quad \alpha_{n-1} = \frac{1 - 1 - 2m_n - ic_n}{\tau_n - 1 + ic_n}, \quad n \geq 1,$$

with $\tau_0 = 1$. Moreover, if the maximal parameter sequence $\{M_n\}_{n=0}^\infty$ of the positive chain sequence $\{d_n\}_{n=1}^\infty$ is different from its minimal parameter sequence $\{m_n\}_{n=0}^\infty$, then the measure μ has a positive mass (pure point) of size M_0 at $z = 1$.

Observe from (2.3) that $\sigma_n = \prod_{k=1}^n (1 + ic_k)$, $n \geq 1$, and that $\sigma_n/\bar{\sigma}_n = 1/\tau_n$, $n \geq 1$. Thus, we can also write

$$\begin{aligned} \sigma_n S_n(z) &= R_n(z) - 2(1 - m_n)R_{n-1}(z), \\ \tau_n \sigma_n S_n^*(z) &= R_n(z) - 2(1 - m_n)zR_{n-1}(z), \quad n \geq 1. \end{aligned} \quad (2.7)$$

With respect to the measure μ the polynomials R_n also satisfy the following so called L orthogonality property

$$\int_{\mathbb{C}} \zeta^{n+j} R_n(\zeta) (1 - \bar{\zeta}) d\mu(\zeta) = 0, \quad 0 \leq j \leq n-1.$$

If we also consider the polynomials $Q_n(z)$, $n \geq 0$, given by

$$Q_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]Q_n(z) - 4d_{n+1}zQ_{n-1}(z), \quad n \geq 1, \quad (2.8)$$

with $Q_0(z) = 0$ and $Q_1(z) = 2d_1$, where $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ are as in (2.4) and (2.6), then we can state the following lemma.

Lemma 2.1.

$$R_n(1) - 2(1 - m_n)R_{n-1}(1) = Q_n(1) - 2(1 - m_n)Q_{n-1}(1) = 2^n m_1 m_2 \dots m_n, \quad n \geq 1.$$

Proof. Easily verified from the three term recurrence relations for R_n and Q_n . \square

Gaussian type quadrature rules on the unit circle are based on the zeros of POPUC. We can derive information about the Gaussian type quadrature rule based on the zeros of $(z - 1)R_n(z)$ from the following results given very recently in [6].

We get

$$\frac{R_n(z) - Q_n(z)}{(z - 1)R_n(z)} = \frac{\lambda_{n,0}}{z - 1} + \sum_{k=1}^n \frac{\lambda_{n,k}}{z - z_{n,k}}, \quad n \geq 1,$$

where $z_{n,k}$, $k = 1, 2, \dots, n$, are the zeros of $R_n(z)$ and the quantities $\lambda_{n,k}$, $k = 0, 1, \dots, n$, shown in [6] to be positive, satisfy

$$\lambda_{n,0} = 1 - \frac{Q_n(1)}{R_n(1)} \quad \text{and} \quad \lambda_{n,k} = \frac{Q_n(z_{n,k})}{(1 - z_{n,k})R'_n(z_{n,k})}, \quad k = 1, 2, \dots, n. \quad (2.9)$$

Moreover,

$$\sum_{k=0}^{\infty} \mu_{k+1} z^k = \frac{R_n(z) - Q_n(z)}{(z - 1)R_n(z)} = O(z^n), \quad n \geq 1,$$

and

$$\sum_{k=0}^{\infty} \mu_k z^{k-1} = \frac{R_n(z) - Q_n(z)}{(z - 1)R_n(z)} = O\left((1/z)^{n+2}\right), \quad n \geq 1,$$

where $\mu_k = \int_{\mathbb{C}} \zeta^k d\mu(\zeta)$, $k = 0, \pm 1, \pm 2, \dots$.

Thus, by the same idea used by Gauss to discover the Gaussian quadrature formulas, we obtain the following.

Theorem 2.2. Let $z_{n,k} = e^{i\theta_{n,k}}$, $k = 1, 2, \dots, n$, be the zeros of R_n and let $z_{n,0} = 1$. Then the following quadrature formula holds.

$$\int_{\mathbb{C}} \zeta^p d\mu(\zeta) = \sum_{k=0}^n \lambda_{n,k} (z_{n,k})^p, \quad p = 0, \pm 1, \dots, \pm n, \quad (2.10)$$

where the weights $\lambda_{n,k}$, $k = 0, 1, \dots, n$, are all positive and can be given in terms of $\{R_n\}_{n=0}^\infty$ and $\{d_n\}_{n=1}^\infty$ by

$$\begin{aligned} \lambda_{n,0} &= \frac{(1 - m_n)2^{2n} d_1 d_2 \dots d_n}{R_n(1)R_n(1) - 2(1 - m_n)R_{n-1}(1)}, \\ \lambda_{n,k} &= \frac{2^{2n-1} d_1 d_2 \dots d_n z_{n,k}^{n-1}}{(1 - z_{n,k})R'_n(z_{n,k})R_{n-1}(z_{n,k})}, \quad k = 1, 2, \dots, n. \end{aligned} \quad (2.11)$$

Proof. From the recurrence formulas for $\{R_n(z)\}_{n=0}^\infty$ and $\{Q_n(z)\}_{n=0}^\infty$ together with the theory of continued fractions

$$\begin{aligned} \frac{Q_n(1)}{R_n(1)} &= \cfrac{d_1}{1} - \cfrac{d_2}{1} - \cfrac{d_3}{1} - \dots - \cfrac{d_n}{1}, \\ &= (1 - m_0) \cfrac{m_1}{1} - \cfrac{(1 - m_1)m_2}{1} - \dots - \cfrac{(1 - m_{n-1})m_n}{1}, \end{aligned}$$

for $n \geq 1$, where $\{m_n\}_{n=0}^\infty$ is the minimal parameter sequence of the positive chain sequence $\{d_n\}_{n=1}^\infty$. Hence, one can write (see the proof of Lemma 3.2 in [7, p. 83])

$$\lambda_{n,0} = 1 - \frac{Q_n(1)}{R_n(1)} = \frac{1}{1 + \sum_{k=1}^n \frac{m_1 m_2 \cdots m_k}{(1-m_1)(1-m_2)\cdots(1-m_k)}}, \quad n \geq 1, \quad (2.12)$$

which shows the positiveness of $\lambda_{n,0}$. The expression for $\lambda_{n,0}$ in the theorem, which we use in Section 4 of this manuscript, can be obtained as follows.

From Lemma 2.1,

$$1 - \frac{Q_n(1)}{R_n(1)} = \frac{Q_n(1)}{R_n(1)} - \frac{2(1-m_n)Q_{n-1}(1)}{2(1-m_n)R_{n-1}(1)} = \frac{Q_n(1)}{R_n(1)} - \frac{2(1-m_n)[Q_n(1)R_{n-1}(1) - Q_{n-1}(1)R_n(1)]}{[R_n(1) - 2(1-m_n)R_{n-1}(1)]R_n(1)},$$

from which using the determinant formulas

$$Q_n(z)R_{n-1}(z) - Q_{n-1}(z)R_n(z) = 2^{2n-1}d_1d_2\cdots d_nz^{n-1}, \quad n \geq 1,$$

the required value for $\lambda_{n,0}$ follows.

Now for the expressions concerning the coefficients $\lambda_{n,k}$, $k \geq 1$, we first write (2.9) as

$$\lambda_{n,k} = \frac{Q_n(z_{n,k})R_{n-1}(z_{n,k}) - Q_{n-1}(z_{n,k})R_n(z_{n,k})}{(1 - z_{n,k})[R'_n(z_{n,k})R_{n-1}(z_{n,k})]}.$$

Thus, the determinant formula gives the required expressions for the coefficients $\lambda_{n,k}$. The positiveness of $\lambda_{n,k}$ has been shown in [6] using the expression

$$\lambda_{n,k} = \frac{Q_n(z_{n,k})R_{n-1}(z_{n,k}) - Q_{n-1}(z_{n,k})R_n(z_{n,k})}{(1 - z_{n,k})[R'_n(z_{n,k})R_{n-1}(z_{n,k}) - R'_{n-1}(z_{n,k})R_n(z_{n,k})]}.$$

This completes the proof of the Theorem. \square

The above quadrature rule is based on the zeros of the para orthogonal polynomial $S_{n+1}(z) = \tau_{n+1}(1)S_{n+1}^*(z)$. Hence, as it was pointed out by one of the referees, it is the same as the so called Szegő–Radau quadrature rule of order $n+1$ with the fixed node at $z = 1$, originally considered in [2] for symmetric measures. For general measures some properties of the quadrature rule based on the zeros of $S_{n+1}(z) = \tau_{n+1}(w)S_{n+1}^*(z)$, referred to as a Szegő–Radau quadrature rule with the fixed node at $z = w$, can be found in [16].

3. Sieved para-orthogonal polynomials

With $\ell \geq 2$, we consider the CD kernel POPUC $(z = 1)R_n^{(\ell)}(z)$ associated with the sieved OPUC $\{S_n^{(\ell)}(z)\}_{n=0}^\infty$ given in Section 1. We have

$$R_n^{(\ell)}(z) = \sigma_n^{(\ell)} \frac{zS_n^{(\ell)}(z) - \tau_n^{(\ell)}S_n^{(\ell)*}(z)}{z - 1}, \quad n \geq 1,$$

where

$$\begin{aligned} \tau_0^{(\ell)} &= \frac{S_0^{(\ell)}(1)}{S_0^{(\ell)*}(1)} = 1 \quad \text{and} \quad \tau_n^{(\ell)} = \frac{S_n^{(\ell)}(1)}{S_n^{(\ell)*}(1)} = \frac{\tau_{n-1}^{(\ell)}\bar{\alpha}_{n-1}^{(\ell)}}{1 - \tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}}, \quad n \geq 1, \\ \sigma_0^{(\ell)} &= 1 \quad \text{and} \quad \sigma_n^{(\ell)} = \frac{1 - \tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}}{1 - \operatorname{Re}(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})} \sigma_{n-1}^{(\ell)}, \quad n \geq 1, \end{aligned} \quad (3.1)$$

As in (2.4), the polynomials $\{R_n^{(\ell)}\}_{n=0}^\infty$ satisfy the recurrence formulas

$$R_{n+1}^{(\ell)}(z) = \left[(1 + ic_{n+1}^{(\ell)})z + (1 - ic_{n+1}^{(\ell)}) \right] R_n^{(\ell)}(z) - 4d_{n+1}^{(\ell)}zR_{n-1}^{(\ell)}(z), \quad n \geq 1, \quad (3.2)$$

with $R_0^{(\ell)}(z) = 1$ and $R_1^{(\ell)}(z) = (1 + ic_1^{(\ell)})z + (1 - ic_1^{(\ell)})$, where the real sequences $\{c_n^{(\ell)}\}_{n=1}^\infty$ and $\{d_{n+1}^{(\ell)}\}_{n=1}^\infty$ are such that

$$\begin{aligned} c_n^{(\ell)} &= \frac{\operatorname{Im}(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})}{1 - \operatorname{Re}(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})} = i \frac{\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}}{\tau_{n-1}^{(\ell)} + \tau_{n-1}^{(\ell)*}}, \\ d_{n+1}^{(\ell)} &= \frac{1}{4} \frac{[1 - |\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}|^2][1 - \tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)}]^2}{[1 - \operatorname{Re}(\tau_{n-1}^{(\ell)}\alpha_{n-1}^{(\ell)})][1 - \operatorname{Re}(\tau_n^{(\ell)}\alpha_n^{(\ell)})]}, \quad n \geq 1. \end{aligned} \quad (3.3)$$

Moreover, $\{d_{n+1}^{(\ell)}\}_{n=1}^\infty$, where $d_{n+1}^{(\ell)} = (1 - m_n^{(\ell)})m_{n+1}^{(\ell)}$, is a positive chain sequence with the parameter sequence $\{m_{n+1}^{(\ell)}\}_{n=0}^\infty$ given by

$$m_{n+1}^{(\ell)} = \frac{1}{2} \frac{|1 - \tau_n^{(\ell)}\alpha_n^{(\ell)}|^2}{[1 - \operatorname{Re}(\tau_n^{(\ell)}\alpha_n^{(\ell)})]}, \quad n \geq 0.$$

Setting $d_1^{(\ell)} = m_1^{(\ell)}$, we can also say that $\{m_n^{(\ell)}\}_{n=0}^\infty$, with $m_0^{(\ell)} = 0$, is the minimal parameter sequence of the positive chain sequence $\{d_n^{(\ell)}\}_{n=1}^\infty$ and that if $\{M_n^{(\ell)}\}_{n=0}^\infty$ is the maximal parameter sequence of $\{d_n^{(\ell)}\}_{n=1}^\infty$, then $M_0^{(\ell)}$ (if $M_0^{(\ell)} > 0$) is the mass size of the pure point at $z = 1$ in the measure $\mu^{(\ell)}$.

From (1.2), since

$$\alpha_{r\ell+j}^{(\ell)} = 0, \quad 0 \leq j \leq \ell - 2, \quad \text{and} \quad \alpha_{r\ell+\ell-1}^{(\ell)} = \alpha_r,$$

for $r \geq 0$, from (3.1) we have

$$\tau_{r\ell+j}^{(\ell)} = \tau_r, \quad 0 \leq j \leq \ell - 1, \quad r \geq 0.$$

Thus, from (2.3) and (3.1),

$$\sigma_{r\ell+j}^{(\ell)} = \sigma_r, \quad 0 \leq j \leq \ell - 1, \quad r \geq 0.$$

Moreover, from (2.5) and (3.3),

$$\begin{aligned} m_{r\ell+j+1}^{(\ell)} &= \frac{1}{2}, \quad 0 \leq j \leq \ell - 2, \quad \text{and} \quad m_{r\ell+\ell}^{(\ell)} = m_{r+1}, \\ c_{r\ell+j+1}^{(\ell)} &= 0, \quad 0 \leq j \leq \ell - 2, \quad \text{and} \quad c_{r\ell+\ell}^{(\ell)} = c_{r+1}, \\ d_{r\ell+1}^{(\ell)} &= \frac{1}{2}(1 - m_r), \quad d_{r\ell+j+1}^{(\ell)} = \frac{1}{4}, \quad 1 \leq j \leq \ell - 2, \quad \text{and} \quad d_{r\ell+\ell}^{(\ell)} = \frac{1}{2}m_{r+1}, \end{aligned}$$

for $r \geq 0$.

Using these results together with $d\mu^{(\ell)}(z) = \ell^{-1}d\mu(z^\ell)$, we can state the following theorem.

Theorem 3.1. Let $\{d_n\}_{n=1}^\infty$ be a positive chain sequence with the minimal parameter sequence $\{m_n\}_{n=0}^\infty$ and the maximal parameter sequence $\{M_n\}_{n=0}^\infty$. For $\ell \geq 2$, set $\{d_n^{(\ell)}\}_{n=1}^\infty$ to be the positive chain sequence with its minimal parameter sequence given by

$$m_{r\ell+j+1}^{(\ell)} = \frac{1}{2}, \quad 0 \leq j \leq \ell - 2, \quad \text{and} \quad m_{(r+1)\ell}^{(\ell)} = m_{r+1} \quad \text{for } r \geq 0.$$

If $g_0 = \ell^{-1}M_0$ and $g_n = d_n^{(\ell)} / (1 - g_{n-1})$, $n \geq 1$, then $\{g_n\}_{n=0}^\infty = \{M_n^{(\ell)}\}_{n=0}^\infty$ is the maximal parameter sequence of $\{d_n^{(\ell)}\}_{n=1}^\infty$.

Since

$$R_{r\ell+j}^{(\ell)}(z) = \sigma_{r\ell+j}^{(\ell)} \frac{z S_{r\ell+j}^{(\ell)}(z)}{z - 1} \frac{\tau_{r\ell+j}^{(\ell)} S_{r\ell+j}^{(\ell)*}(z)}{z - 1}, \quad 0 \leq j \leq \ell - 1, \quad r \geq 0,$$

we obtain from (1.3), (1.4) that

$$R_{r\ell+j}^{(\ell)}(z) = \sigma_r \frac{z^{j+1} S_r(z^\ell) - \tau_r S_r^*(z^\ell)}{z - 1}, \quad 0 \leq j \leq \ell - 1, \quad r \geq 0. \quad (3.4)$$

With the above results we can state the following.

Theorem 3.2. Let the real sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ be such that $\{d_n\}_{n=1}^\infty$ is also a positive chain sequence with its minimal parameter sequence denoted by $\{m_n\}_{n=0}^\infty$. Let the polynomials $\{R_n(z)\}_{n=0}^\infty$ be given by the three term recurrence formula

$$R_{n+1}(z) = [(1 + ic_{n+1})z + (1 - ic_{n+1})]R_n(z) - 4d_{n+1}zR_{n-1}(z), \quad n \geq 1,$$

with $R_0(z) = 1$ and $R_1(z) = (1 + ic_1)z + (1 - ic_1)$. Then for any $r \geq 1$, $\ell \geq 1$ and $0 \leq j \leq \ell - 1$, the polynomial $R_{r\ell+j}^{(\ell)}(z)$, of degree $r\ell + j$, given by

$$R_{r\ell+j}^{(\ell)}(z) = \frac{z^{j+1}}{z - 1} R_r(z^\ell) + 2(1 - m_r) \frac{z^\ell}{z - 1} \frac{z^{j+1}}{1} R_{r-1}(z^\ell), \quad (3.5)$$

has $r\ell + j$ simple zeros on the unit circle $|z| = 1$ with $z \neq 1$. Moreover, if $N = n + 1$, then between any two zeros of $R_N^{(\ell)}$ (on the unit circle) there is a zero of $R_n^{(\ell)}$.

Proof. The expression for $R_{r\ell+j}^{(\ell)}(z)$ is immediate taking into account (2.7) in (3.4). Properties of the zeros of these polynomials follow from the three term recurrence formula (3.2) and results given in [10]. \square

Note that for $\ell = 1$ the polynomials $R_n^{(\ell)}(z)$ become the polynomials $R_n(z)$. For $\ell \geq 2$, considering only the polynomials $R_{r\ell+\ell-1}^{(\ell)}$ we have

$$\begin{aligned} R_{\ell-1}^{(\ell)}(z) &= \frac{z^\ell S_0(z^\ell) - \tau_0 S_0^*(z^\ell)}{z - 1} \frac{z^\ell - 1}{z - 1}, \\ R_{r\ell-1}^{(\ell)}(z) &= \frac{\prod_{j=0}^{r-2} [1 - \tau_j z_j]}{\prod_{j=0}^{r-2} [1 - \text{Re}(\tau_j z_j)]} \frac{z^\ell S_r(z^\ell) - \tau_r 1 S_{r-1}^*(z^\ell)}{z - 1} \frac{z^\ell - 1}{z - 1} R_{r-1}(z^\ell), \quad r \geq 2. \end{aligned} \quad (3.6)$$

Thus, we also have the following three term recurrence formula

$$R_{(r+2)\ell-1}^{(\ell)}(z) = [(1 + ic_{r+1})z^\ell + (1 - ic_{r+1})]R_{(r+1)\ell-1}^{(\ell)}(z) - 4d_{r+1}z^\ell R_{r\ell-1}^{(\ell)}(z), \quad r \geq 1,$$

with

$$R_{\ell-1}^{(\ell)}(z) = \frac{z^\ell}{z-1} \quad \text{and} \quad R_{2\ell-1}^{(\ell)}(z) = \frac{z^\ell}{z-1} [(1+ic_1)z^\ell + (1-ic_1)].$$

4. Some remarks on sieved Gaussian quadrature

As given by [Theorem 2.2](#), the quadrature rules based on the zeros of the sieved para-orthogonal polynomials $(z-1)R_n^{(\ell)}(z)$ are

$$\int_C \zeta^p d\mu^{(\ell)}(\zeta) = \sum_{m=0}^n \lambda_{n,m}^{(\ell)} (z_{n,m}^{(\ell)})^p, \quad p = 0, \pm 1, \dots, \pm n, \quad (4.1)$$

where $z_{n,0}^{(\ell)} = 1, z_{n,m}^{(\ell)}, m = 1, 2, \dots, n$, are the zeros of $R_n^{(\ell)}(z)$, and the positive weights $\lambda_{n,m}^{(\ell)}, m = 0, 1, \dots, n$, can be given by

$$\begin{aligned} \lambda_{n,0}^{(\ell)} &= \frac{(1-m_n^{(\ell)})2^{2n}d_1^{(\ell)}d_2^{(\ell)}\dots d_n^{(\ell)}}{R_n^{(\ell)}(1)[R_n^{(\ell)}(1)-2(1-m_n^{(\ell)})R_{n-1}^{(\ell)}(1)]}, \\ \lambda_{n,m}^{(\ell)} &= \frac{2^{2n-1}d_1^{(\ell)}d_2^{(\ell)}\dots d_n^{(\ell)}(z_{n,m}^{(\ell)})^{n-1}}{(1-z_{n,m}^{(\ell)})R_n^{(\ell)}(z_{n,m}^{(\ell)})R_{n-1}^{(\ell)}(z_{n,m}^{(\ell)})}, \quad m = 1, 2, \dots, n. \end{aligned} \quad (4.2)$$

However, we can say a bit more about the Gaussian quadrature formula based on the zeros of $(z-1)R_{r\ell+\ell-1}^{(\ell)}(z), r \geq 1$. We write this quadrature formula in the form

$$\int_C \zeta^p d\mu^{(\ell)}(\zeta) = \sum_{k=0}^r \sum_{j=0}^{\ell-1} \rho_{r,k,j} (w_{r,k,j})^p,$$

where $w_{r,0,0} = z_{r\ell+\ell-1,0}^{(\ell)} = 1$ and the remaining $w_{r,k,j} = z_{r\ell+\ell-1,k\ell+j}^{(\ell)}$ are the zeros of $R_{r\ell+\ell-1}^{(\ell)}(z)$ and $\rho_{r,k,j} = \lambda_{r\ell+\ell-1,k\ell+j}^{(\ell)}$, $j = 0, 1, \dots, \ell-1, k = 0, 1, \dots, r$.

Theorem 4.1. *The nodes $w_{r,k,j}$ and the weights $\rho_{r,k,j}$ of the quadrature formula based on the zeros of the polynomial $(z-1)R_{r\ell+\ell-1}^{(\ell)}(z)$ are such that*

$$\begin{aligned} w_{r,0,j} &= e^{i2\pi j/\ell}, \quad j = 0, 1, \dots, \ell-1, \\ w_{r,k,j} &= e^{i(\theta_{r,k}+2\pi j)/\ell}, \quad j = 0, 1, \dots, \ell-1, k = 1, 2, \dots, r, \end{aligned}$$

and

$$\rho_{r,k,j} = \frac{1}{\ell} \lambda_{r,k}, \quad j = 0, 1, \dots, \ell-1, k = 0, 1, \dots, r.$$

Here, $z_{r,k} = e^{i\theta_{r,k}}$ and $\lambda_{r,k}$ are as in [Theorem 2.2](#).

Proof. The formulas for the nodes simply follow from [\(3.6\)](#). The results for weights can be verified in two different ways. The first one is the direct substitution of the zeros in the expressions given by [\(4.2\)](#). For example, using information such as $(w_{r,0,j})^\ell = 1$ and $(w_{r,k,j})^\ell = z_{r,k}$, we have

$$\begin{aligned} d_1^{(\ell)} \dots d_{\ell-1}^{(\ell)} d_\ell^{(\ell)} d_{\ell+1}^{(\ell)} \dots d_{(r-1)\ell+\ell-1}^{(\ell)} d_{r\ell} d_{r\ell+1} \dots d_{r\ell+\ell-1}^{(\ell)} &= \frac{1}{2} (1-m_0) \dots \frac{1}{4} \frac{1}{2} m_1 \frac{1}{2} (1-m_1) \dots \frac{1}{4} \frac{1}{2} m_r \frac{1}{2} (1-m_r) \dots \frac{1}{4}, \\ &= \frac{2}{4^{(r+1)\ell-1-r}} (1-m_r) d_1 d_2 \dots d_r, \end{aligned}$$

$$\begin{aligned} (1-w_{r,0,j}) R_{r\ell+\ell-1}^{(\ell)'}(w_{r,0,j}) &= \ell (w_{r,0,j})^{\ell-1} R_r(1), \\ (1-w_{r,k,j}) R_{r\ell+\ell-1}^{(\ell)'}(w_{r,k,j}) &= \ell (1-z_{r,k}) (w_{r,k,j})^{\ell-1} R_r'(z_{r,k}). \end{aligned}$$

and

$$\begin{aligned} R_{r\ell+\ell-2}^{(\ell)}(w_{r,0,j}) &= (w_{r,0,j})^{-1} [R_r(1) - 2(1-m_r)R_{r-1}(1)], \\ R_{r\ell+\ell-2}^{(\ell)}(w_{r,k,j}) &= 2(1-m_r)(w_{r,k,j})^{\ell-1} R_{r-1}(z_{r,k}). \end{aligned}$$

Thus the relations $\rho_{r,k,j} = \ell^{-1} \lambda_{r,k}$ follow from [\(2.11\)](#) and [\(4.2\)](#).

The other way to verify these results is to obtain the new quadrature formula [\(4.1\)](#) directly from the quadrature formula [\(2.10\)](#). Since $(w_{r,k,j})^\ell = z_{r,k}$, we have from [\(2.11\)](#), with $n = r$,

$$\int_{2\pi j/\ell}^{2\pi(j+1)/\ell} (e^{i\ell\theta})^p d\mu(e^{i\ell\theta}) = \sum_{k=0}^r \lambda_{r,k} (w_{r,k,j})^{\ell p}, \quad p = 0, \pm 1, \dots, \pm r,$$

for $j = 0, 1, \dots, \ell - 1$. Hence,

$$\int_0^{2\pi} (e^{i\ell\theta})^p d\mu(e^{i\ell\theta}) = \sum_{j=0}^{\ell-1} \sum_{k=0}^r \lambda_{r,k} (w_{r,k,j})^{\ell p}, \quad p = 0, \pm 1, \dots, \pm r,$$

As a consequence, we can write

$$\int_0^{2\pi} (e^{i\theta})^{\ell p} d\mu^{(\ell)}(e^{i\theta}) = \sum_{k=0}^r \left[\ell^{-1} \lambda_{r,k} \sum_{j=0}^{\ell-1} (w_{r,k,j})^{\ell p} \right], \quad p = 0, \pm 1, \dots, \pm r.$$

Note that the weights and nodes are exactly what we wanted. Only thing we still need to verify is the validity of the above quadrature rule for the powers $\ell p + q$, $q = 1, 2, \dots, \ell - 1$.

The left hand side can be written as

$$\int_0^{2\pi} (e^{i\theta})^{\ell p + q} d\mu^{(\ell)}(e^{i\theta}) = \sum_{j=0}^{\ell-1} \int_0^{2\pi/\ell} (e^{i\theta + i2\pi j/\ell})^{\ell p + q} d\mu^{(\ell)}(e^{i\theta}) = \int_0^{2\pi/\ell} (e^{i\theta})^{\ell p + q} \left[\sum_{j=0}^{\ell-1} (e^{i2\pi j/\ell})^q \right] d\mu^{(\ell)}(e^{i\theta}).$$

For $q = 1, 2, \dots, \ell - 1$, since the sum within the integral is zero the resulting integral is zero.

Since $w_{r,k,j}$ are the ℓ^{th} roots of $z_{r,k}$, it follows that

$$\sum_{j=0}^{\ell-1} (w_{r,k,j})^{\ell p + q} = \sum_{j=0}^{\ell-1} e^{i(\theta_{r,k} + 2\pi j)(\ell p + q)/\ell} = 0, \quad q = 1, 2, \dots, \ell - 1.$$

So the right hand side is also zero for such powers. \square

Acknowledgment

The authors would like to thank the referees for the many valuable comments.

References

- [1] J. Breuer, E. Ryckman, B. Simon, Equality of the spectral and dynamical definitions of reflection, *Commun. Math. Phys.* 295 (2010) 531–550.
- [2] A. Bultheel, L. Darius, P. González-Vera, A connection between quadrature formulas on the unit circle and the interval $[-1, 1]$, *J. Comput. Appl. Math.* 132 (2001) 1–14.
- [3] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad, Quadrature and orthogonal rational functions, *J. Comput. Appl. Math.* 127 (2001) 67–91.
- [4] M.J. Cantero, L. Moral, L. Velázquez, Measures and para-orthogonal polynomials on the unit circle, *East J. Approx.* 8 (2002) 447–464.
- [5] K. Castillo, L. Garza, F. Marcellán, Perturbations on the subdiagonals of Toeplitz matrices, *Linear Algebra Appl.* 434 (2011) 1563–1579.
- [6] K. Castillo, M.S. Costa, A. Sri Ranga, D.O. Veronese, A Favard type theorem for orthogonal polynomials on the unit circle from a three term recurrence formula, *J. Approx. Theory* 184 (2014) 146–162.
- [7] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and its Applications Series, Gordon and Breach, New York, 1978.
- [8] M.S. Costa, H.M. Felix, A. Sri Ranga, Orthogonal polynomials on the unit circle and chain sequences, *J. Approx. Theory* 173 (2013) 14–32.
- [9] M.S. Costa, E. Godoy, R.L. Lamblém, A. Sri Ranga, Basic hypergeometric functions and orthogonal Laurent polynomials, *Proc. Am. Math. Soc.* 140 (2012) 2075–2089.
- [10] D.K. Dimitrov, A. Sri Ranga, Zeros of a family of hypergeometric para-orthogonal polynomials on the unit circle, *Math. Nachr.* 286 (2013) 1778–1791.
- [11] L. Golinskii, Quadrature formula and zeros of para-orthogonal polynomials on the unit circle, *Acta Math. Hungarica* 96 (2002) 169–186.
- [12] P. González-Vera, J.C. Santos-León, O. Njåstad, Some results about numerical quadrature on the unit circle, *Adv. Comput. Math.* 5 (1996) 297–328.
- [13] W.B. Gragg, Positive definite Toeplitz matrices, the Arnoldi process for isometric operators, and Gaussian quadrature on the unit circle, *J. Comput. Appl. Math.* 46 (1993) 183–198. E.S. Nikolaev (Ed.), Russian original in *Numerical Methods in Linear Algebra*, Moscow University Press, 1982, pp. 16–32.
- [14] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, vol. 98, Cambridge Univ. Press, Cambridge, 2005.
- [15] M.E.H. Ismail, Xin Li, On sieved orthogonal polynomials IX: orthogonality on the unit circle, *Pacific J. Math.* 152 (1992) 289–297.
- [16] C. Jagels, L. Reichel, Szegő–Lobatto quadrature rules, *J. Comput. Appl. Math.* 200 (2007) 116–126.
- [17] W.B. Jones, O. Njåstad, W.J. Thron, Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, *Bull. Lond. Math. Soc.* 21 (1989) 113–152.
- [18] A. Kheifets, L. Golinskii, F. Peherstorfer, P. Yuditskii, Scattering Theory for CMV matrices: uniqueness, Helson–Szegő and strong Szegő theorems, *Integral Equ. Oper. Theory* 69 (2011) 479–508.
- [19] F. Marcellán, G. Sansigre, Orthogonal polynomials on the unit circle: symmetrization and quadratic decomposition, *J. Approx. Theory* 65 (1991) 109–119.
- [20] F. Marcellán, G. Sansigre, Symmetrization, quadratic decomposition and cubic transformations of orthogonal polynomials on the unit circle, in: C. Brezinski, L. Gori, A. Ronveaux (Eds.), *Orthogonal polynomials and their applications* (Erice, 1990), IMACS Ann. Comput. Appl. Math., vol. 9, Baltzer, Basel, 1991, pp. 341–345.
- [21] F. Peherstorfer, Positive trigonometric quadrature formulas and quadrature on the unit circle, *Math. Comput.* 80 (2011) 1685–1701.
- [22] F. Peherstorfer, A. Volberg, P. Yuditskii, CMV matrices with asymptotically constant coefficients. Szegő–Blaschke class, scattering theory, *J. Funct. Anal.* 256 (2009) 2157–2210.
- [23] B. Simanek, Week convergence of CD kernels: a new approach on the circle and real line, *J. Approx. Theory* 164 (2012) 204–209.
- [24] B. Simon, Orthogonal polynomials on the unit circle. Part 1. Classical theory, *Am. Math. Soc. Colloq. Publ.*, vol. 54, part 1, Am. Math. Soc., Providence, RI, 2005.

- [25] B. Simon, Orthogonal polynomials on the unit circle. Part 2. Spectral theory, Am. Math. Soc. Colloq. Publ., vol. 54, part 2, Am. Math. Soc., Providence, RI, 2005.
- [26] B. Simon, Szegő's Theorem and its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials, Princeton Univ. Press, Princeton, 2011.
- [27] A. Sri Ranga, Szegő polynomials from hypergeometric functions, Proc. Am. Math. Soc. 138 (2010) 4259–4270.
- [28] G. Szegő, Orthogonal Polynomials, Am. Math. Soc. Colloq. Publ., fourth ed., vol. 23, Am. Math. Soc., Providence, RI, 1975.
- [29] S. Tsujimoto, A. Zhedanov, Elliptic hypergeometric Laurent biorthogonal polynomials with a dense point spectrum on the unit circle, SIGMA Symmetry Integrability Geom. Methods Appl. 5 (2009). 30p.
- [30] M.L. Wong, First and second kind paraorthogonal polynomials and their zeros, J. Approx. Theory 146 (2007) 282–293.